A strong formulation for the graph partition problem

Sunil Chopra* Sangho Shim†

October 1, 2019

Abstract

We develop a polynomial size extended graph formulation of the graph partition problem which dominates the formulation introduced by Chopra and Rao [2], and show that the extended graph formulation is tight on a tree. We introduce exponentially many valid inequalities to the Chopra-Rao formulation, which we call generalized arc inequalities, and develop a linear time algorithm to separate the most violated generalized arc inequality. We show that the polynomial size extended graph formulation is equivalent to the Chopra-Rao formulation augmented by the exponentially many generalized arc inequalities.

Keywords—integer programming; combinatorial optimization; extended formulation; network clustering; graph partition problem; extended graph formulation; primal dual construction

1 Introduction

Given a graph $G = (V, E)$ with edge weights c_e for $e \in E$ and $k \geq 0$, the graph partition problem is to partition the nodes V into k labeled subsets N_1, ..., N_k such that $N_i \cap N_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{r} N_i = V$ to minimize the total weight of the edges with end points in two different subsets, *i.e.*, the edges cut. We allow subsets N_i in the partition to be empty. In this paper, we use two terms *subsets* and *clusters* interchangeably. This problem is obviously NP-hard since MAX-CUT is a special case of the problem (see [6]). A graph partitioning may be regarded as a graph coloring where each node is colored with exactly one of k colors. That is, a partition $\Pi = (N_1, N_2, ..., N_k)$ is equivalent to a node coloring where the nodes in N_t are colored with *color* $t \in K = \{1, 2, ..., k\}$.

The partition problem has been studied by several authors. In the context of scheduling Carlson and Nemhauser [1] consider the problem of partitioning a graph into at most k subsets with no restriction on the size of each subset. Grötschel and Wakabayashi [7, 8, 9] have studied the problem when G is a complete graph and is to be partitioned into at most $|V|$ subsets. They have called it the clique partitioning problem. There is no restriction on the number of nodes in each subset.

In this paper, we provide an extended graph formulation for the graph partition problem with a polynomial number of variables and constraints and show that its LP relaxation has only integer extreme points when G is a tree. We also develop the generalized arc inequalities and show that the formulation introduced by Chopra and Rao [2] along with the (exponentially many) generalized arc inequalities provides an LP relaxation that has only integer extreme points when G is a tree. The partition problem when G is a tree arises in many contexts such as the allocation of computer information to blocks of storage (see Lukes [11]) and the convex recoloring problem in phylogenetic and linguistics (see Moran and Snir [14]).

*Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208, USA,

†School of Engineering, Mathematics and Science, Robert Morris University, Moon Twp, PA 15090, USA, `shim@rmu.edu`

A partition $\Pi = (N_1, ..., N_k)$ of the nodes V is encoded into the incidence vector (x, y) where $x_{it} = 1$ if node $i \in N_t$ and $y_e = 1$ if edge e is not cut by the partition. We will use x_{it} and $x(i,t)$ interchangeably. We refer to y_e here as an *edge variable*. We will use y_{uw} and $y(u,w)$ interchangeably for an edge $uw \in E(G)$. Chopra and Rao [2] showed that the incidence vectors of all partitions satisfy the following constraints:

$$(x, y) \geq 0, \tag{1}$$

$$\sum_{t=1}^{k} x_{vt} = 1 \qquad \text{for all nodes } v, \tag{2}$$

$$x_{ut} + x_{wt} - y_{uw} \leq 1 \qquad \text{for all edges } uw \text{ and all } t = 1, ..., k, \tag{3}$$

$$\left.\begin{array}{r} x_{ut} - x_{wt} + y_{uw} \leq 1 \\ -x_{ut} + x_{wt} + y_{uw} \leq 1 \end{array}\right\} \qquad \text{for all edges } uw \text{ and all } t = 1, ..., k. \tag{4}$$

We refer to (2) as *partitioning equations*, (3) as *edge inequalities* and the pair of inequalities in (4) as *arc inequalities*. The edge inequalities imply that the edge is not cut if its end nodes are in the same subset. The arc inequalities imply its converse; the end nodes of an edge are in the same subset if the edge is not cut.

Define $LP1(G, k) = \{(x, y) : (x, y) \text{ satisfies (1)-(4)}\}$. If $P1(G, k)$ is the convex hull of the incidence vectors of r partitions for $r \leq k$, Chopra and Rao [2] showed that $P1(G, k) = conv\{(x, y) \in LP1(G, k) : x, y \text{ integer}\}$. While the constraints (1)-(4) provide the basis of an integer programming formulation for the graph partition problem, the LP relaxation defined by $LP1(G, k)$ is very weak and does not completely describe $P1(G, k)$ even for the simple case when G is only an edge (u, w) and $k \geq 4$. The fractional solution

$$(x_{u1} = x_{u2} = x_{w3} = x_{w4} = y_{uw} = 1/2) \tag{5}$$

with all other variables 0 is an extreme point of $LP1(uw, 4)$. Chopra and Rao [2] introduced several classes of facet defining inequalities for graphs with cycles. We now provide a set of inequalities that generalize (4), are facet defining for $P1(G, k)$, and are violated by the fractional solution (5). These new inequalities apply even when G is a tree. Assume that $k \geq 3$. For each edge uw and a proper subset S of $\{1, ..., k\}$, we define a pair of *generalized arc inequalities* (GAI)

$$\left.\begin{array}{r} \sum_{t \in S} x_{ut} - \sum_{t \in S} x_{wt} + y_{uw} \leq 1 \\ -\sum_{t \in S} x_{ut} + \sum_{t \in S} x_{wt} + y_{uw} \leq 1 \end{array}\right\} \tag{6}$$

Observe that (4) is a special case of (6) with $|S| = 1$. We first observe that (5) violates a generalized arc inequality in (6) for $S = \{1, 2\}$. We define a stronger LP relaxation by adding the generalized arc inequalities (6) to $LP1(G, k)$. Define $LQ1(G, k) = \{(x, y) : (x, y) \text{ satisfies (1), (2), (3) and (6)}\}$.

We may rewrite (6) as

$$\left.\begin{array}{r} \sum_{t \in S}(x_{ut} - x_{wt}) + y_{uw} \leq 1 \\ -\sum_{t \in S}(x_{ut} - x_{wt}) + y_{uw} \leq 1 \end{array}\right\} \tag{7}$$

Since there are exponentially many proper subsets $S \subset \{1, ..., k\}$, the generalized arc inequalities are exponentially many. However, we can identify the most violated generalized arc inequality (7) against a solution $(\hat{x}, \hat{y})$ within linear time by using

$$\begin{aligned} S(u, w) &= \{t : \hat{x}_{ut} - \hat{x}_{wt} > 0\}, \\ S(w, u) &= \{t : \hat{x}_{wt} - \hat{x}_{ut} > 0\}. \end{aligned}$$

The inequalities (1)-(4) are all shown in Chopra and Rao [2] to be facet defining for $P1(G, k)$. In the same manner, the generalized arc inequalities are also facet defining for $P1(G, k)$:

Proposition 1. *The generalized arc inequalities (6) are facet defining for $P1(G, k)$.*

For $LQ1(G, k)$, at least exponentially many generalized arc inequalities are required. We precisely state the proposition as follows (for a proof, the readers may refer to Appendix B):

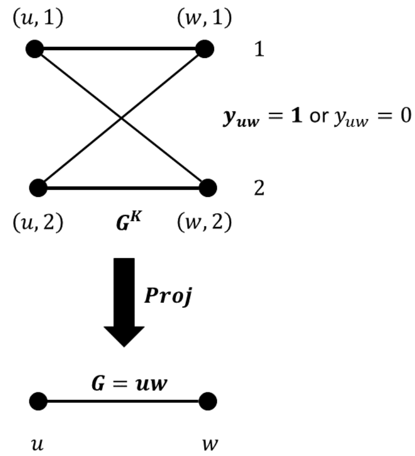

Figure 1: The 2-fold lifted graph G^k of the edge graph $G = uw$

Proposition 2. *Let $G = (V, E)$ be a graph, and let $k \geq 3$. Then the system of (1)-(3) and (6) for the non-empty subsets $S = \{1, ..., k-1\}$ is a minimal description of $LQ1(G, k)$.*

The rest of the paper is organized as follows. In Section 2 we provide an extended graph formulation for the graph partition problem with a polynomial number of variables and constraints. We show that the LP relaxation of the extended graph formulation is at least as strong as $LQ1(G, k)$. In other words, the constraints in the extended graph formulation imply the inequalities (2), (3) and (6). In Section 3, we show that the LP relaxation of the extended graph formulation has only integer extreme points when G is a tree. In Section 4, we show that $LQ1(G, k)$ is equivalent to the LP relaxation of the extended graph formulation for any graph G. This implies that $LQ1(G, k)$ also has only integer extreme points when G is a tree.

2 An extended graph formulation for the graph partition problem

To develop an extended graph formulation of the graph partition problem, we first construct an extended graph $G^k = (V^k, E^k)$ given the original graph $G = (V, E)$ and the number of subsets k. We replace each node u in V by k nodes $(u, 1), (u, 2), ..., (u, k)$ in V^k. Thus, $V^k = \{(u, t) \text{ for all } u \text{ in } V \text{ and } 1 \leq t \leq k\}$. We replace each edge (u, w) in E by k^2 edges $((u, t_u), (w, t_w))$ for $1 \leq t_u \leq k$ and $1 \leq t_w \leq k$. Thus, $E^k = \{((u, t_u), (w, t_w)) \text{ for all } (u, w) \in E, 1 \leq t_u \leq k \text{ and } 1 \leq t_w \leq k\}$. Observe that each edge in G is replaced by a $k \times k$ complete bipartite graph in G^k. Figure 1 shows the extended graph G^2 for the edge graph $G = (u, w)$ and $k = 2$.

An edge $((u, t_u), (w, t_w))$ of G^k can be used to indicate that the node u is assigned to subset t_u and the node w is assigned to subset t_w. Thus, assigning the node u to subset t_u and the node w to subset t_w in G is equivalent to selecting the edge $((u, t_u), (w, t_w))$ from the bipartite graph in G^k corresponding to the edge (u, w). For example, in Figure 1, thick edges $((u, 1), (w, 1))$ and $((u, 2), (w, 2))$ indicate partitions $(N_1 = \{u, w\}, N_2 = \emptyset)$ and $(N_1 = \emptyset, N_2 = \{u, w\})$ respectively which lead to $y_{uw} = 1$. Thin edges $((u, 1), (w, 2))$ and $((u, 2), (w, 1))$ indicate partitions $(N_1 = \{u\}, N_2 = \{w\})$ and $(N_1 = \{w\}, N_2 = \{u\})$ respectively which lead to $y_{uw} = 0$.

In general, a partition of G can be represented by a copy of G in the extended graph G^k. For example, consider $G = C_3 = u - v - w - u$ to be the triangle shown in Figure 2. The graph $(u, 1) - (v, 1) - (w, 2) - (u, 1)$ in G^2 corresponds to a copy of G and the partition $(N_1 = \{u, v\}, N_2 = \{w\})$. It is one of the eight possible copies of the triangle G in G^2: $(u, t_u) - (v, t_v) - (w, t_w) - (u, t_u)$ with $t_u, t_v, t_w \in K = \{1, 2\}$.

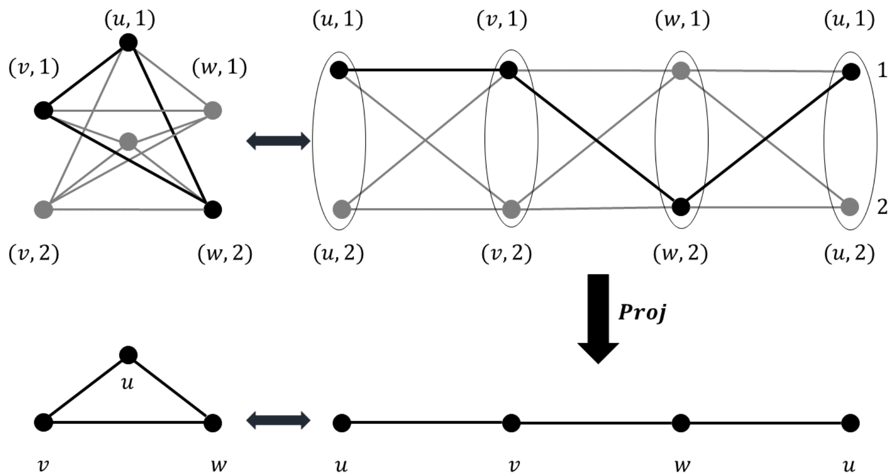

Figure 2: In the 2-fold lifted graph G^k of the cycle $G = C_3 = u - v - w - u$ of three nodes u, v, w, a copy $(u,1) - (v,1) - (w,2) - (u,1)$ of G indicates a partition $(N_1 = \{u, v\}, N_2 = \{w\})$.

Now we develop a formulation for the graph partition problem on the extended graph and refer to it as the *extended graph formulation*. Given the graph $G = (V, E)$ and k, the first step is to construct the extended graph $G^k = (V^k, E^k)$. For each edge $((u, t_u), (w, t_w))$ in E^k define the variable $z((u, t_u), (w, t_w))$. In an incidence vector corresponding to a partition, this variable takes the value 1 if the edge $((u, t_u), (w, t_w))$ is selected (implying that node u is assigned to set t_u and node w is assigned to set t_w) and 0 if it is not. Define $K = \{1, 2, ..., k\}$. Incidence vectors in the extended graph formulation corresponding to partitions satisfy the following constraints:

$$z((u, t_u), (w, t_w)) \geq 0 \text{ for } uw \in E(G) \text{ and } t_u, t_w \in K, \tag{8}$$

$$\sum_{t_u \in K} \sum_{t_w \in K} z((u, t_u), (w, t_w)) = 1 \text{ for } uw \in E(G), \tag{9}$$

$$\sum_{t_v \in K} z((v, t_v), (u, t_u)) = \sum_{t_w \in K} z((u, t_u), (w, t_w)) \text{ for any 2-path } v - u - w \text{ and for } t_u \in K. \tag{10}$$

In fact, we don't need (10) for all 2-paths. Let d be the degree of node u in G and let $\{v, w_1, ..., w_{d-1}\}$ be the neighborhood of u. Then, we only need $d - 1$ equations

$$\sum_{t_v \in K} z((v, t_v), (u, t_u)) = \sum_{t_j \in K} z((u, t_u), (w_j, t_j)), j \in \{1, ..., d-1\},$$

for (u, t_u), and we don't need any of

$$\sum_{t_i \in K} z((u, t_u), (w_i, t_i)) = \sum_{t_j \in K} z((u, t_u), (w_j, t_j)), i \neq j \in \{1, ..., d-1\}.$$

Thus, the number of constraints is $O(|E|k)$.

We define an extended formulation $\overline{ELP1}(G, k)$ of $LP1(G, k)$ by (x, y, z) satisfying (1)-(4) and (8)-(10)

along with linking constraints

$$y(u,w) = \sum_{t \in K} z((u,t),(w,t)) \tag{11}$$

$$\left.\begin{aligned} x(u,t_u) &= \textstyle\sum_{t_w \in K} z((u,t_u),(w,t_w)) \text{ for } t_u \in K \\ x(w,t_w) &= \textstyle\sum_{t_u \in K} z((u,t_u),(w,t_w)) \text{ for } t_w \in K \end{aligned}\right\} \tag{12}$$

for each edge $(u,w) \in E$.

In Section 2.1, we show that (1)-(4) are induced by (8)-(12). (Moreover, the generalized arc inequalities (6) are all induced.) In particular, the linking constraints (12) are well-defined by (10). In the remaining system (8)-(12) of $\overline{ELP1}(G,k)$, (x,y)-variables in (11)-(12) are dependent variables which are decided by z-variables satisfying (8)-(10). Without dependent variables (x,y), we define the LP relaxation $ELP1(G,k) = \{z : z \text{ satisfies (8)-(10)}\}$ and the integer hull $EP1(G,k) = \{z \in ELP1(G,k), z \text{ integer}\}$ of the extended graph formulation. The extended graph formulation $ELP1(G,k)$ is thus equivalent to the extended formulation $\overline{ELP1}(G,k)$. In this regard, (11) and (12) project z-variables of $ELP1(G,k)$ to the (x,y)-variables of $LP1(G,k)$ (and $LQ1(G,k)$). In the remaining of this paper, we regard the extended graph formulation $ELP1(G,k)$ as an extended formulation of $LP1(G,k)$.

2.1 The extended graph formulation is at least as strong as the Chopra-Rao formulation with generalized arc inequalities

We show for a general graph G that the extended graph formulation is at least as strong as Chopra-Rao formulation strengthened by the generalized arc inequalities; *i.e.*, $Proj(ELP1(G)) \subseteq LQ1(G)$. We do so by showing that $(x,y) \in Proj(ELP1(G))$ satisfies the inequalities (2), (3) and (6).

Theorem 3. *$Proj(ELP1(G))$ is contained in $LQ1(G)$. Thus, each $(x,y) \in Proj(ELP1(G))$ satisfies the partitioning equations (2), the edge inequalities (3), and the generalized arc inequalities (6).*

Proof. Given $z \in ELP1(G)$, consider $(x,y) \in Proj(ELP1(G))$. Thus, (x,y) satisfies (11) and (12). For a node $v \in V$ and an adjacent node $w \in V$ in G, using (12) we thus have

$$\sum_{t \in K} x(v,t) = \sum_{t \in K} \sum_{\bar{t} \in K} z\left((v,t),(w,\bar{t})\right).$$

Using (9) we thus obtain

$$\sum_{t \in K} x(v,t) = \sum_{t \in K} \sum_{\bar{t} \in K} z\left((v,t),(w,\bar{t})\right) = 1,$$

which implies that (x,y) satisfies the partitioning equations (2). Now consider an edge $(u,w) \in E$. Using

(11) and (12) we thus have

$$\begin{aligned}
&x(u,t)+x(w,t)-y(u,w)\\
&= \sum_{t_w\in K} z\left((u,t),(w,t_w)\right)+\sum_{t_u\in K} z\left((u,t_u),(w,t)\right)-\sum_{t_y\in K} z\left((u,t_y),(w,t_y)\right)\\
&= \sum_{t_w\in K} z\left((u,t),(w,t_w)\right)+\sum_{t_u\in K} z\left((u,t_u),(w,t)\right)-\Big(z\left((u,t),(w,t)\right)\Big)\\
&\quad -\sum_{t_y\in K\setminus\{t\}} z\left((u,t_y),(w,t_y)\right)\\
&= \left(\sum_{t_w\in K} z\left((u,t),(w,t_w)\right)+\sum_{t_u\in K} z\left((u,t_u),(w,t)\right)-z\left((u,t),(w,t)\right)\right)\\
&\quad -\sum_{t_y\in K\setminus\{t\}} z\left((u,t_y),(w,t_y)\right)\\
&\le \left(\sum_{t_u\in K}\sum_{t_w\in K} z\left((u,t_u),(w,t_w)\right)\right)-0=1-0=1
\end{aligned}$$

Thus, (x,y) satisfies the edge inequalities (3).

Next we show that (x,y) satisfies the generalized arc inequalities (4) as follows: For any edge $(u,w)\in E$ and $S\subset K$, using (11) and (12) we obtain

$$\begin{aligned}
&\sum_{t\in S} x(u,t)-\sum_{t\in S} x(w,t)+y(u,w)\\
&= \sum_{t\in S}\sum_{t_w\in K} z\left((u,t),(w,t_w)\right)-\sum_{t\in S}\sum_{t_u\in K} z\left((u,t_u),(w,t)\right)+\sum_{t_y\in K} z\left((u,t_y),(w,t_y)\right)\\
&= \sum_{t\in S}\sum_{t_w\in K} z\left((u,t),(w,t_w)\right)+\left(-\sum_{t\in S}\sum_{t_u\in K\setminus\{t\}} z\left((u,t_u),(w,t)\right)\right)+\sum_{t_y\in K\setminus S} z\left((u,t_y),(w,t_y)\right)\\
&= \sum_{t\in S}\sum_{t_w\in K} z\left((u,t),(w,t_w)\right)+\sum_{t_y\in K\setminus S} z\left((u,t_y),(w,t_y)\right)+\left(-\sum_{t\in S}\sum_{t_u\in K\setminus\{t\}} z\left((u,t_u),(w,t)\right)\right)\\
&\le \sum_{t_u\in K}\sum_{t_w\in K} z\left((u,t_u),(w,t_w)\right)+\Big(-0\Big)=1.
\end{aligned}$$

In the same manner, we can show that

$$-\sum_{t\in S} x(u,t)+\sum_{t\in S} x(w,t)+y(u,w)\le 1.$$

Thus, (x,y) satisfies the generalized arc inequalities (6). □

Theorem 3 shows that the extended graph formulation is at least as strong as the Chopra-Rao formulation augmented with all generalized arc inequalities. This is particularly interesting because the extended graph formulation has a polynomial number of variables and constraints whereas there are exponentially many generalized arc inequalities.

3 The extended graph formulation is tight for a tree

In this section, we prove that $ELP1(G,k)$ is tight on a tree G. To prove it, we follow the approach suggested by Martin, Rardin and Campbell [13].

Let G be a tree and fix a root node of the tree. We number the root node as $n = |V(G)|$. The notion of *arborescence* directs the edges heading for the root. We denote by $\vec{G}$ the tree with the edges directed to the root. We can number the non-root nodes $v = 1, 2, ..., n-1$ such that an arc $(u, w) \in E(\vec{G})$ is directed from u to w if $u < w$. Linking dummy root ∞ to the root n by an arc (n, ∞) directed from n to ∞, we have an augmented rooted tree $\vec{G}_\infty$. For every node $w = 1, ..., n$, there is only one arc (w, w') going out from w in $\vec{G}_\infty$.

Linking $(\infty, \emptyset)$ to $(n, t), t \in K$, we construct the augmented extended graph $\vec{G}^k_\infty$. The directions of the arcs (u, w) of $\vec{G}_\infty$ are extended to those of $\vec{G}^k_\infty$: For every arc $(u, w) \in E\left(\vec{G}\right)$ directed from u to w, the extended graph arcs of $\vec{G}^k_\infty$ are directed from (u, t_u) to (w, t_w) for $t_u, t_w \in K$; *i.e.*,

$$((u, t_u), (w, t_w)) \in E\left(\vec{G}^k\right) \text{ for } (u, w) \in E\left(\vec{G}\right) \text{ and for } t_u, t_w \in K.$$

The augmented edges $((n, t), (\infty, \emptyset)) \in E\left(\vec{G}^k_\infty\right)$ are directed from (n, t) to $(\infty, \emptyset)$ for $t \in K$.

Employing dummy variables $x(n, t), t \in K$ corresponding to the augmented arcs $((n, t), (\infty, \emptyset))$ of $\vec{G}^k_\infty$, we rewrite the primal problem over $ELP1(G, k)$ on a tree G as follows:

$$\min \sum_{((u,t_u),(w,t_w))\in \vec{E}(G^k)} c\left((u, t_u), (w, t_w)\right) \cdot z\left((u, t_u), (w, t_w)\right) \tag{13}$$

$$s.t. \quad \sum_{t\in K} x(n, t) = 1, \tag{14}$$

$$\sum_{t_v \in K} z((v, t_v), (n, t)) - x(n, t) = 0 \text{ for } (v, n) \in E\left(\vec{G}\right) \text{ and for } t \in K, \tag{15}$$

$$\left.\begin{array}{r} \displaystyle\sum_{t_v\in K} z((v, t_v), (u, t_u)) - \sum_{t_w \in K} z((u, t_u), (w, t_w)) = 0 \\ \text{for 2-path } v \to u \to w \text{ and for } t_u \in K \end{array}\right\} \tag{16}$$

$$x(n, t) \geq 0 \text{ for } t \in K, \tag{17}$$

$$z\left((u, t_u), (w, t_w)\right) \geq 0 \text{ for } \left((u, t_u), (w, t_w)\right) \in E\left(\vec{G}^k\right). \tag{18}$$

Note that (14)-(16) imply (9). A primal integer feasible solution indicates a copy of $\vec{G}_\infty$ in $\vec{G}^k_\infty$.

We employ the dual variable ν corresponding to (14). The dual variable ν is the dual objective value. For $(v, n) \in E\left(\vec{G}\right)$ and $t \in K$, the dual variables corresponding to constraints (15) are denoted by $\mu(v, (n, t), \infty)$. The first two components $(v, (n, t))$ of $\mu(v, (n, t), \infty)$ indicate the extended graph arcs $((v, t_v), (n, t)) \in E\left(\vec{G}^k_\infty\right)$ coming into (n, t). The last two components $((n, t), \infty)$ of $\mu(v, (n, t), \infty)$ indicate $x(n, t)$ or the extended graph arc $((n, t), (\infty, \emptyset))$ going out from (n, t). Since there is only one arc $(n, \infty) \in E\left(\vec{G}_\infty\right)$ going out from root n, we may leave out the third component and denote dual variable $\mu(v, (n, t), \infty)$ simply by $\mu(v, (n, t))$. Likewise, the dual variables corresponding to (16) are denoted by $\mu(v, (u, t_u))$, as (u, w) is the only arc going out from u and w may be left out.

The dual problem is

$$\max \quad \nu \tag{19}$$

$$s.t. \quad \nu - \sum_{(v,n)\in \vec{E}(G)} \mu(v, (n, t)) \leq 0 \text{ for } t \in K, \tag{20}$$

$$\left.\begin{array}{r} \displaystyle\mu(u, (w, t_w)) - \left(\sum_{(u',u)\in E(\vec{G})} \mu(u', (u, t_u))\right) \leq c\left((u, t_u), (w, t_w)\right) \\ \text{for } \left((u, t_u), (w, t_w)\right) \in E\left(\vec{G}^k\right) \end{array}\right\} \tag{21}$$

where (20) and (21) correspond to $x(n,t)$ and $z\left((u,t_u),(w,t_w)\right)$.

Martin, Rardin and Campbell [13] solved the shortest path problem on a directed acyclic hypergraph by a dynamic programming algorithm which proves the integrality of an ILP formulation of the problem using the dual and the primal constructions. The nodes of the directed acyclic hypergraph are totally ordered such that the directions of the hyper arcs are from smaller numbered nodes to larger numbered nodes. The dual construction in dynamic programming algorithm enumerates the nodes in the forward manner (in increasing order of the nodes) to the global state (the largest numbered node) for the dual optimal value. Then, the primal construction keeps track of the process in the backward manner producing the primal integer optimal solution of the same value. This process of the dual and the primal constructions works for every primal objective function and proves the integrality of the ILP formulation. Following their way, we develop a dynamic programming algorithm.

Our dynamic programming algorithm identifies the dual optimal value in DUAL CONSTRUCTION which scans the arcs $E\left(\vec{G}^k\right)$ in the forward manner from the leaf nodes to the root n. In fact, the dynamic programming constructs the minimum subtrees rooted at $(n,t), t\in K$ and selects the dual optimal value among the values of the subtrees. While performing DUAL CONSTRUCTION, PRIMAL CONSTRUCTION records the backward arcs and then keeps track of the backward arcs in the backward manner starting from (n,t_n^*) where the optimal subtree is rooted. The primal integer solution produced by PRIMAL CONSTRUCTION has the same value as the dual optimal value produced by DUAL CONSTRUCTION. In general, DUAL CONSTRUCTION and PRIMAL CONSTRUCTION work for every primal objective function c. It proves that $ELP1(G,k)$ is tight on a tree:

Theorem 4. *Let G be a tree. Then, $ELP1(G,k)=EP1(G,k)$.*

DUAL CONSTRUCTION A dynamic programming algorithm produces the dual optimal value.

- Scan the nodes $w=1,...,n$ in the ascending order. For $(u,w)\in E(\vec{G})$ and $t_w\in K$, set

$$\mu^*\left(u,(w,t_w)\right)=\min_{t_u\in K}\left\{\left(\sum_{(u',u)\in E(\vec{G})}\mu^*\left(u',(u,t_u)\right)\right)+c\left((u,t_u),(w,t_w)\right)\right\}$$

  where (w,w') is the only arc going out from w.

- Fix optimal value

$$\nu^*=\min_{t\in K}\left\{\sum_{(v,n)\in E(\vec{G})}\mu^*\left(v,(n,t)\right)\right\}.$$

PRIMAL CONSTRUCTION In fact, the dynamic programming algorithm constructs k subtrees rooted at $(n,t), t\in K$ in $\vec{G}^k$ and selects the optimal subtree rooted at (n,t_n^*) at the end of DUAL CONSTRUCTION. The corresponding primal construction proceeds in the opposite direction to determine an integer solution z^* as follows:

- While performing the dual construction above, record the backward arcs; *i.e.*, set $t_u(w,t_w)=t_u^*$ such that

$$\mu^*\left(u,(w,t_w)\right)=\left(\sum_{(u',u)\in E(\vec{G})}\mu^*\left(u',(u,t_u^*)\right)\right)+c\left((u,t_u^*),(w,t_w)\right).$$

- Identify t_n^* such that

$$\nu^*=\sum_{(v,n)\in E(\vec{G})}\mu^*\left(v,(n,t_n^*)\right),$$

  and set $x(n,t_n^*)=1$.

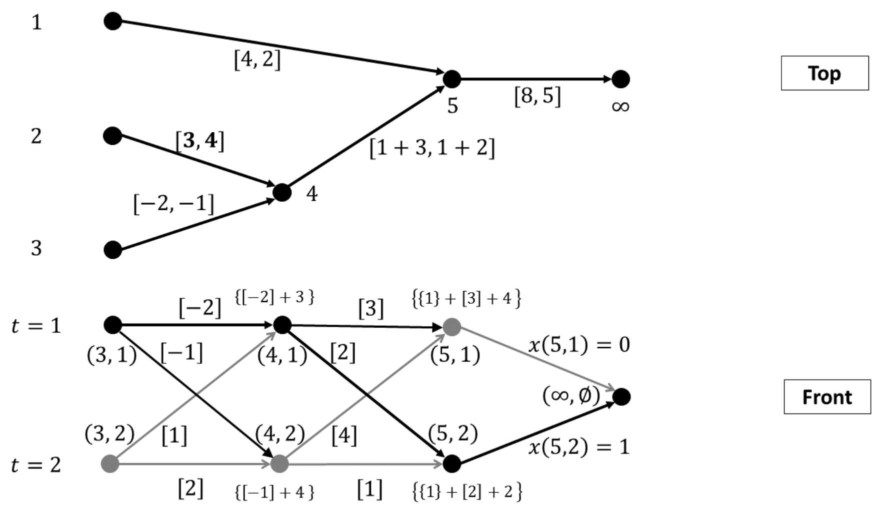

Figure 3: The top and the front views of $\vec{G}^k_\infty$

- Keep track of the backward arcs

$$((u, t_u(w, t_w)), (w, t_w)) \in E\left(\vec{G}^k\right)$$

in the backward manner starting from (n, t_n^*), and indicate them by

$$z^*((u, t_u(w, t_w)), (w, t_w)) = 1.$$

Set the remaining $z^*((u, t_u), (w, t_w)) = 0$.

- Return the primal integer feasible solution z^*.

Figure 3 illustrates the dynamic programming algorithm. The numbers in the brackets on the arcs in the front view are the cost $c((u, t_u), (w, t_w))$ of the arcs. Scanning the arcs $((3, t_3), (4, t_4))$, the minimum costs are

$$\begin{aligned} \mu^*(3, (4, 1)) &= c((3, 1), (4, 1)) = -2 \\ \mu^*(3, (4, 2)) &= c((3, 1), (4, 2)) = -1. \end{aligned}$$

The pair of the numbers ($[\mathbf{3}, \mathbf{4}]$) in bold are

$$[3 = \mu^*(2, (4, 1)), 4 = \mu^*(2, (4, 2))].$$

Then,

$$\begin{aligned} \mu^*(4, (5, 1)) &= \mu^*(3, (4, 1)) + \mu^*(2, (4, 1)) + c((4, 1), (5, 1)) = -2 + 3 + 3 \\ \mu^*(4, (5, 2)) &= \mu^*(3, (4, 1)) + \mu^*(2, (4, 1)) + c((4, 1), (5, 2)) = -2 + 3 + 2. \end{aligned}$$

At the root $n = 5$,

$$\nu^* = \mu^*(4, (5, 2)) + \mu^*(1, (5, 2)) = (1 + 2) + 2 = 5.$$

In the backward manner, path $(3, 1) \to (4, 1) \to (5, 2)$ becomes a part of the primal optimal integer solution.

$ELP1(G, k)$ can have fractional extreme points if the graph G contains a cycle. Figure 4 illustrates the non-zero variables of a fractional extreme point of $ELP1(G, k)$ where G is a 3-cycle defined on the nodes u, v, and w with $k = 2$. Thus, Theorem 4 is the strongest possible result we can obtain for the LP relaxation of the extended graph formulation.

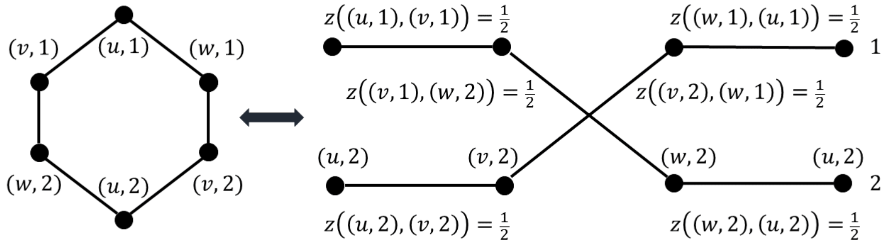

Figure 4: An extreme fractional solution in G^2 over 3-cycle $G = u - v - w - u$

4 The extended graph formulation is equivalent to the strengthened Chopra-Rao formulation

Given a general graph G, we show that for every $(x, y) \in LQ1(G)$, there is $z \in ELP1(G)$ such that $Proj(z) = (x, y)$. This would show that $LQ1(G) \subseteq Proj(ELP1(G))$. We only need to show this result for the edge graph $G = uw$, as both of $LQ1(G)$ and the projection (11)-(12) are defined for each edge. For an example of a 2-path $v - u - w$, (12) on uv and uw implies (10); *i.e.*,

$$x(u, t_u) = \sum\nolimits_{t_v \in K} z((v, t_v), (u, t_u)) = \sum\nolimits_{t_w \in K} z((u, t_u), (w, t_w)).$$

We first illustrate the result for $k = 2$.

4.1 The case $k = 2$

For the case where $k = 2$, we have a formula to obtain $z \in ELP1(G)$ given any $(x, y) \in LQ1(G)$ such that $Proj(z) = (x, y)$. Without loss of generality, we assume that G is the edge graph uw. Given $(x, y) \in P1(G, k = 2)$, define z variables as follows,

$$\begin{aligned} z\left((u,1),(w,1)\right) &= \frac{1}{2}\left(1 + y(u,w) - x(u,2) - x(w,2)\right), && (22)\\ z\left((u,2),(w,2)\right) &= \frac{1}{2}\left(1 + y(u,w) - x(u,1) - x(w,1)\right), && (23)\\ z\left((u,1),(w,2)\right) &= \frac{1}{2}\left(1 - y(u,w) - x(u,1) + x(w,1)\right) && (24)\\ &= \frac{1}{2}\left(1 - y(u,w) + x(u,2) - x(w,2)\right), && (25)\\ z\left((u,2),(w,1)\right) &= \frac{1}{2}\left(1 - y(u,w) - x(u,2) + x(w,2)\right) && (26)\\ &= \frac{1}{2}\left(1 - y(u,w) + x(u,1) - x(w,1)\right). && (27) \end{aligned}$$

Observe that $Proj(z) = (x, y)$ and z satisfies (9). All the z variables are non-negative using the edge inequalities and arc inequalities. Note that each of $z\left((u,1),(w,2)\right)$ and $z\left((u,2),(w,1)\right)$ can be defined in two different ways. That is, (24) is same as (25) by partitioning equations (2). In a similar manner, (26) is the same as (27).

4.2 The cases $k \geq 3$

In this section, for $k \geq 3$ we show that for any $q = (x^q, y^q_{uw}) \in LQ1(G, k)$, there exists $z \in ELP1(G, k)$ such that $Proj(z) = q = (x^q, y^q_{uw})$. We consider the cases $y^q_{uw} = 0$ and $y^q_{uw} > 0$ separately. The two cases are discussed in detail in Sections 4.2.1 and 4.2.2 respectively. Here we summarize the general approach.

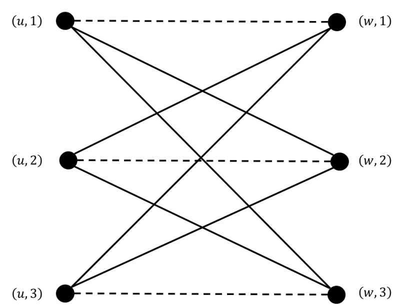

Figure 5: In $y^q_{uw} > 0$ case, we take out y^q_{uw} corresponding to dashed edges so that the remaining components may satisfy $(x', y'_{uw})/(1 - y^q_{uw}) \in LQ1(G = uw, k)$ with $y'_{uw} = 0$.

For the case $y^q_{uw} = 0$, our proof begins with

$$z^{(0)}\left((u, t_u), (w, t_w)\right) = x^q_{ut_u} x^q_{wt_w} \text{ for } t_u, t_w \in K, \tag{28}$$

and iteratively updates $z^{(i-1)}$ to $z^{(i)}$, $i = 1, 2, ...,$ until

$$y^{(i)}_{uw} := \sum_{t \in K} z^{(i)}\left((u, t), (w, t)\right) \tag{29}$$

becomes $y^q_{uw} = 0$. For every i, $z^{(i)}$ will satisfy

$$\left.\begin{aligned} x^q_{ut} = x^{(i)}_{ut} &= \sum_{t_w \in K} z^{(i)}\left((u, t), (w, t_w)\right), \forall t \in K, \\ x^q_{wt} = x^{(i)}_{wt} &= \sum_{t_u \in K} z^{(i)}\left((u, t_u), (w, t)\right), \forall t \in K, \end{aligned}\right\} \tag{30}$$

and $\left(x^{(i)}, y^{(i)}_{uw}\right)$ will satisfy (1)-(4) and (6).

We then consider the case where $y^q_{uw} > 0$. We subtract $z((u,t),(w,t)) = \alpha_t y^q_{uw}$ from x^q_{ut} and x^q_{wt} for $t = 1, ..., k$ so that the remaining components $x'_{ut} = x^q_{ut} - z((u,t),(w,t))$ and $x'_{wt} = x^q_{wt} - z((u,t),(w,t))$ satisfy

$$q^0 = (x^0, y^0_{uw}) = \frac{(x', y'_{uw} = 0)}{1 - y^q_{uw}} \in LQ1(G, k)$$

with $y^0_{uw} = 0$ and there exists a $z^0 \geq 0$ such that $Proj(z^0) = (x^0, y^0_{uw})$. We scale z^0 back to

$$z((u, t_u), (w, t_w)) = z^0((u, t_u), (w, t_w)) \cdot (1 - y^q_{uw}) \text{ for } t_u \neq t_w.$$

In Figure 5, the dashed edges correspond to the z-values of

$$y^q_{uw} = z((u,1),(w,1)) + z((u,2),(w,2)) + z((u,3),(w,3))$$

which can be determined so that the remaining components, $x'_{ut} = x^q_{ut} - z((u,t),(w,t))$ and $x'_{wt} = x^q_{wt} - z((u,t),(w,t))$ for $t = 1, 2, 3$, satisfy

$$(x^0, y^0_{uw}) = \frac{(x', y'_{uw})}{1 - y^q_{uw}} \in LQ1(G, 3) \text{ with } y^0_{uw} = 0.$$

There exists $z^0 \geq 0$ such that $Proj(z^0) = (x^0, y^0)$. We scale z^0 back to

$$z((u, t_u), (w, t_w)) = z^0((u, t_u), (w, t_w)) \cdot (1 - y^q_{uw}) \text{ for } t_u \neq t_w \in \{1, 2, 3\}.$$

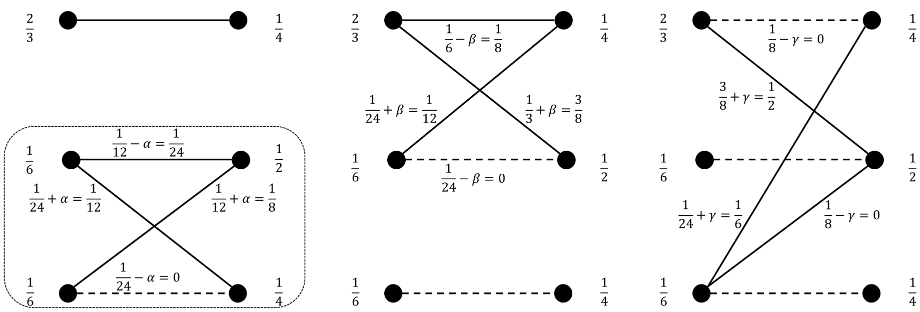

Figure 6: Lifting procedure $z^{(0)} \xrightarrow{\text{Step 1: } \alpha=1/24} z^{(1)} \xrightarrow{\text{Step 2: } \beta=1/24} z^{(2)} \xrightarrow{\text{Step 3: } \gamma=1/8} z^{(3)}$

4.2.1 Zero-valued edge variable

In this section, we show that for $q = (x^q, y^q_{uw}) \in LQ1(G,k)$ with $y^q_{uw} = 0$, there exists $z \in ELP1(G,k)$ such that $q = Proj(z)$:

Theorem 5. *For $k \geq 3$, let $G = uw$ be the edge graph. Let $(x^q, y^q_{uw}) \in LQ1(G,k)$ with $y^q_{uw} = 0$. Then, there is a non-negative solution z to (9) satisfying (11)-(12) with $(x,y) = (x^q, y^q)$.*

To construct z with $q = Proj(z)$, we perform a lifting procedure in three steps. Figure 6 depicts an example of the 3-step procedure on

$$q = (x^q_{u1} = 2/3, x^q_{u2} = 1/6, x^q_{u3} = 1/6; x^q_{w1} = 1/4, x^q_{w2} = 1/2, x^q_{w3} = 1/4; y^q_{uw} = 0).$$

We first define $z^{(0)}$ according to (28). It satisfies (1)-(4) and (6):

Lemma 6. *The initial vector $(x^{(0)}, y^{(0)})$ defined by (29) and (30) satisfies (1)-(4) and (6).*

Proof. The non-negativity constraints (1) and the partition equations (2) are trivial.

We show validity of the edge inequalities (3). Without loss of generality, we only need to show it in the case of $t = 1$. The first equations in (30) for $i = 0$ are satisfied because the partition equations (2) with $t = 1$ imply

$$\begin{aligned} x^{(0)}_{u1} &= \sum_{t_w=1}^{k} x^q_{u1} x^q_{wt_w} = x^q_{u1} \left(\sum_{t_w=1}^{k} x^q_{wt_w} \right) = x^q_{u1} \cdot 1 = x^q_{u1}, \\ x^{(0)}_{w1} &= \sum_{t_u=1}^{k} x^q_{ut_u} x^q_{w1} = \left(\sum_{t_u=1}^{k} x^q_{ut_u} \right) x^q_{w1} = 1 \cdot x^q_{w1} = x^q_{w1}. \end{aligned}$$

The edge inequality holds as follows:

$$\begin{aligned} x^{(0)}_{u1} + x^{(0)}_{w1} - y^{(0)}_{uw} &= x^q_{u1} + x^q_{w1} - \sum_{t \in K} x^q_{ut} x^q_{wt} \\ &= x^q_{u1} + x^q_{w1} - x^q_{u1} x^q_{w1} - \sum_{t \neq 1} x^q_{ut} x^q_{wt} \\ &= 1 + (-1 + x^q_{u1} + x^q_{w1} - x^q_{u1} x^q_{w1}) - \sum_{t \neq 1} x^q_{ut} x^q_{wt} \\ &= 1 - (1 - x^q_{u1})(1 - x^q_{w1}) - \sum_{t \neq 1} x^q_{ut} x^q_{wt} \\ &\leq 1 + 0 + 0 = 1. \end{aligned}$$

For $S \subseteq K$, we show a generalized arc inequality (6) as follows:

$$\begin{aligned}
&\sum_{t\in S} x_{ut}^{(0)} - \sum_{t\in S} x_{wt}^{(0)} + y_{uw}^{(0)} \\
&= \left(\sum_{t\in S}\left(x_{ut}^{(0)} - x_{wt}^{(0)}\right)\right) + y_{uw}^{(0)} = \left(\sum_{t\in S}\left(x_{ut}^{q} - x_{wt}^{q}\right)\right) + \sum_{t\in K} x_{ut}^{q} x_{wt}^{q} \\
&= \left(\sum_{t\in S}\left(x_{ut}^{q} - x_{wt}^{q}\right)\right) + \sum_{t_u\in K}\sum_{t_w\in K} x_{ut_u}^{q} x_{wt_w}^{q} - \sum_{t_u\neq t_w} x_{ut_u}^{q} x_{wt_w}^{q} \\
&= \left(\sum_{t\in S}\left(x_{ut}^{q}\cdot(1) - (1)\cdot x_{wt}^{q}\right)\right) + \left(\sum_{t_u\in K} x_{ut_u}^{q}\right)\cdot\left(\sum_{t_w\in K} x_{wt_w}^{q}\right) - \sum_{t_u\neq t_w} x_{ut_u}^{q} x_{wt_w}^{q}.
\end{aligned}$$

By partitioning equations (2),

$$\begin{aligned}
&\sum_{t\in S} x_{ut}^{(0)} - \sum_{t\in S} x_{wt}^{(0)} + y_{uw}^{(0)} \\
&= \left(\sum_{t\in S}\left(x_{ut}^{q}\cdot\left(\sum_{t_w\in K} x_{wt_w}^{q}\right) - \left(\sum_{t_u\in K} x_{ut_u}^{q}\right)\cdot x_{wt}^{q}\right)\right) + (1)\cdot(1) - \sum_{t_u\neq t_w} x_{ut_u}^{q} x_{wt_w}^{q} \\
&= \left(\sum_{t\in S}\left(x_{ut}^{q}\cdot\left(\sum_{t_w\in K\setminus\{t\}} x_{wt_w}^{q}\right) - \left(\sum_{t_u\in K\setminus\{t\}} x_{ut_u}^{q}\right)\cdot x_{wt}^{q}\right)\right) + 1 - \sum_{t_u\neq t_w} x_{ut_u}^{q} x_{wt_w}^{q} \\
&= \left(\sum_{t\in S}\sum_{t_w\in K\setminus\{t\}} x_{ut}^{q} x_{wt_w}^{q} - \sum_{t\in S}\sum_{t_u\in K\setminus\{t\}} x_{ut_u}^{q} x_{wt}^{q}\right) - \sum_{t_u\neq t_w} x_{ut_u}^{q} x_{wt_w}^{q} + 1 \\
&= -\left(\sum_{t_u\neq t_w} x_{ut_u}^{q} x_{wt_w}^{q} - \sum_{t\in S}\sum_{t_w\in K\setminus\{t\}} x_{ut}^{q} x_{wt_w}^{q}\right) - \sum_{t\in S}\sum_{t_u\in K\setminus\{t\}} x_{ut_u}^{q} x_{wt}^{q} + 1 \leq 0 + 0 + 1 = 1.
\end{aligned}$$

□

Once $z^{(0)}$ satisfies (1)-(4) and (6), $z^{(i)}$ will also satisfy them for every iteration i of the following three step procedure. By changing the order of indices $t \in K$, we may assume without loss of generality that

$$z^{(0)}\left((u,1),(w,1)\right) = x_{u1}^{q} x_{w1}^{q} \geq x_{ut}^{q} x_{wt}^{q} = z^{(0)}\left((u,t),(w,t)\right) \text{ for } t \neq 1.$$

Step 1: The first step focuses on indices $t \in K \setminus \{1\}$. If $z^{(0)}((u,1),(w,1)) = 0$, $z = z^{(0)}$ satisfies $q = Proj(z)$. If $z^{(0)}((u,1),(w,1)) > 0$ and $z^{(0)}((u,t),(w,t)) = 0$ for $t \in K \setminus \{1\}$, set $i \leftarrow 1$ and go to Step 3. If there is only one $t \in K \setminus \{1\}$ with non-zero value $z^{(0)}((u,t),(w,t)) > 0$, we assume $z^{(0)}((u,2),(w,2)) > 0$ and $x_{ut}^{q} x_{wt}^{q} = 0$ for $t > 2$, set $i \leftarrow 1$ and go to Step 2. If there are at least two indices $t \in K \setminus \{1\}$ with non-zero values $z^{(0)}((u,t),(w,t)) > 0$, set $i \leftarrow 1$ and repeat the following procedure to make one of the non-zero values vanish until at most one $t \in K \setminus \{1\}$ allows non-zero value $z^{(i)}((u,t),(w,t)) > 0$.

Let a pair of indices $t_1 \neq t_2 \in K\setminus\{1\}$ have non-zero values $z^{(i-1)}((u,t_1),(w,t_1)) > 0$ and $z^{(i-1)}((u,t_2),(w,t_2)) > 0$. We make one of the non-zero values vanish by

$$\left.\begin{aligned}
z^{(i)}\left((u,t_1),(w,t_1)\right) &= z^{(i-1)}\left((u,t_1),(w,t_1)\right) - \alpha_{i-1} \\
z^{(i)}\left((u,t_2),(w,t_2)\right) &= z^{(i-1)}\left((u,t_2),(w,t_2)\right) - \alpha_{i-1} \\
z^{(i)}\left((u,t_1),(w,t_2)\right) &= z^{(i-1)}\left((u,t_1),(w,t_2)\right) + \alpha_{i-1} \\
z^{(i)}\left((u,t_2),(w,t_1)\right) &= z^{(i-1)}\left((u,t_2),(w,t_1)\right) + \alpha_{i-1}
\end{aligned}\right\} \tag{31}$$

where $\alpha_{i-1} = \min\left\{z^{(i-1)}\left((u,t_1),(w,t_1)\right), z^{(i-1)}\left((u,t_2),(w,t_2)\right)\right\}$. The other variables remain without change; *i.e.*,

$$z^{(i)}\left((u,t_u),(w,t_w)\right) \;\; = \;\; z^{(i-1)}\left((u,t_u),(w,t_w)\right) \text{ for } t_u, t_w \in K \setminus \{t_1, t_2\}.$$

One of $z^{(i)}\left((u,t_1),(w,t_1)\right), z^{(i)}\left((u,t_2),(w,t_2)\right)$ becomes zero. If $z^{(i)}\left((u,t),(w,t)\right) = 0$ for $t \neq 1$, set $i \leftarrow i+1$ and go to Step 3. If there is only one $t \in K \setminus \{1\}$ with non-zero value $z^{(i)}\left((u,t),(w,t)\right) > 0$, we assume $z^{(i)}\left((u,2),(w,2)\right) > 0$ and $x^q_{ut}x^q_{wt} = 0$ for $t > 2$, set $i \leftarrow i+1$ and go to Step 2. If there are at least two indices $t \in K \setminus \{1\}$ with non-zero values $z^{(i)}\left((u,t),(w,t)\right) > 0$, set $i \leftarrow i+1$ and repeat the procedure (31) to make one of the non-zero-values vanish until at most one $t \in K \setminus \{1\}$ allows a non-zero value $z^{(i)}\left((u,t),(w,t)\right) \neq 0$.

For every iteration $i \geq 1$ in this first step,

$$\begin{aligned}
&\sum_{t_u \neq 1} \sum_{t_w \neq 1} z^{(i-1)}\left((u,t_u),(w,t_w)\right) \\
&\quad = \sum_{t_u \neq 1} \sum_{t_w \neq 1} z^{(i-1)}\left((u,t_u),(w,t_w)\right) + \left(\alpha_{i-1} + \alpha_{i-1} - \alpha_{i-1} - \alpha_{i-1}\right) \\
&\quad = \sum_{t_u \neq 1} \sum_{t_w \neq 1} z^{(i)}\left((u,t_u),(w,t_w)\right),
\end{aligned}$$

and therefore the sum over $t_u, t_w \in K \setminus \{1\}$ is constant:

$$\sum_{t_u \neq 1} \sum_{t_w \neq 1} z^{(i)}\left((u,t_u),(w,t_w)\right) = \sum_{t_u \neq 1} \sum_{t_w \neq 1} z^{(0)}\left((u,t_u),(w,t_w)\right) = \sum_{t_u \neq 1} \sum_{t_w \neq 1} x^q_{ut_u} x^q_{wt_w}, \tag{32}$$

where the right-hand side is given by (28).

Step 2: In Step 1, we do not update $z((u,1),(w,1))$; *i.e.*,

$$z^{(i-1)}((u,1),(w,1)) = z^{(0)}((u,1),(w,1)) = x^q_{u1}x^q_{w1}.$$

From our assumption, we have $z^{(i-1)}\left((u,1),(w,1)\right) \geq z^{(i-1)}\left((u,2),(w,2)\right)$ and $z^{(i-1)}\left((u,t),(w,t)\right) = 0$ for $t > 2$. This second step changes $z^{(i-1)}\left((u,t_u),(w,t_w)\right)$ for $t_u, t_w \in \{1,2\}$ as follows:

$$\left.\begin{aligned}
z^{(i)}\left((u,1),(w,1)\right) &= \;\; z^{(i-1)}\left((u,1),(w,1)\right) - \beta \\
z^{(i)}\left((u,2),(w,2)\right) &= \;\; z^{(i-1)}\left((u,2),(w,2)\right) - \beta \\
z^{(i)}\left((u,1),(w,2)\right) &= \;\; z^{(i-1)}\left((u,1),(w,2)\right) + \beta \\
z^{(i)}\left((u,2),(w,1)\right) &= \;\; z^{(i-1)}\left((u,2),(w,1)\right) + \beta
\end{aligned}\right\}$$

where $\beta = \min\left\{z^{(i-1)}\left((u,1),(w,1)\right), z^{(i-1)}\left((u,2),(w,2)\right)\right\} = z^{(i-1)}\left((u,2),(w,2)\right)$. The other variables remain unchanged. Observe that

$$\sum_{t_u \neq 1} \sum_{t_w \neq 1} z^{(i)}\left((u,t_u),(w,t_w)\right) = -\beta + \sum_{t_u \neq 1} \sum_{t_w \neq 1} z^{(i-1)}\left((u,t_u),(w,t_w)\right) = -\beta + \sum_{t_u \neq 1} \sum_{t_w \neq 1} x^q_{ut_u} x^q_{wt_w}. \tag{33}$$

Now, we made $z^{(i-1)}\left((u,2),(w,2)\right)$ vanish to be $z^{(i)}\left((u,2),(w,2)\right) = 0$. If $z^{(i)}\left((u,1),(w,1)\right) = 0$, $z = z^{(i)}$ satisfies $q = Proj(z)$. If $z^{(i)}\left((u,1),(w,1)\right) > 0$, set $i \leftarrow i+1$ and go to Step 3.

Step 3: We have $z^{(i-1)}\left((u,1),(w,1)\right) = x^q_{u1}x^q_{v1} - \beta > 0$, where $\beta = 0$ from Step 1 or $\beta = z^{(i-2)}\left((u,2),(w,2)\right)$ from Step 2, and $z^{(i-1)}\left((u,t),(w,t)\right) = 0$ for $t \neq 1$. Edge inequality $x^q_{u1} + x^q_{w1} = x^q_{u1} + x^q_{w1} + y^q_{uw} \leq 1$ implies that

$$\begin{aligned}
z^{(i-1)}((u,1),(w,1)) &= x^q_{u1}x^q_{w1} - \beta = (x^q_{u1}x^q_{w1} - \beta) + (0) \\
&\le (x^q_{u1}x^q_{w1} - \beta) + (1 - x^q_{u1} - x^q_{w1}) = -\beta + 1 - x^q_{u1} - x^q_{w1} + x^q_{u1}x^q_{w1} \\
&= -\beta + (1 - x^q_{u1})(1 - x^q_{w1}) = -\beta + \left(\sum_{t\neq 1} x^q_{ut}\right)\left(\sum_{t\neq 1} x^q_{wt}\right) \\
&= -\beta + \sum_{t_u\neq 1}\sum_{t_w\neq 1} x^q_{ut_u}x^q_{wt_w}.
\end{aligned}$$

Using (32) if coming to Step 3 from Step 1 ($\beta = 0$), or using (33) if coming from Step 2, we obtain

$$\begin{aligned}
-\beta + \sum_{t_u\neq 1}\sum_{t_w\neq 1} x^q_{ut_u}x^q_{wt_w} &= \sum_{t_u\neq 1}\sum_{t_w\neq 1} z^{(i-1)}((u,t_u),(w,t_w)) \\
&= \sum_{t_u\neq t_w\in K\setminus\{1\}} z^{(i-1)}((u,t_u),(w,t_w)) + \sum_{t\in K\setminus\{1\}} z^{(i-1)}((u,t),(w,t)) \\
&= \sum_{t_u\neq t_w\in K\setminus\{1\}} z^{(i-1)}((u,t_u),(w,t_w)) + 0 = \sum_{t_u\neq t_w\in K\setminus\{1\}} z^{(i-1)}((u,t_u),(w,t_w)).
\end{aligned}$$

Since $z^{(i-1)}((u,1),(w,1)) \le \sum\left\{z^{(i-1)}((u,t_u),(w,t_w)) : t_u \neq t_w \in K\setminus\{1\}\right\}$, this third step can pick a non-zero value $z^{(i-1)}((u,t_u),(w,t_w)) > 0, t_u \neq t_w \in K\setminus\{1\}$ and reduce $z^{(i-1)}((u,1),(w,1))$ by $\gamma_{i-1} = \min\{z^{(i-1)}((u,1),(w,1)), z^{(i-1)}((u,t_u),(w,t_w))\}$ as follows:

$$\left.\begin{aligned}
z^{(i)}((u,1),(w,1)) &= z^{(i-1)}((u,1),(w,1)) - \gamma_{i-1} \\
z^{(i)}((u,t_u),(w,t_w)) &= z^{(i-1)}((u,t_u),(w,t_w)) - \gamma_{i-1} \\
z^{(i)}((u,1),(w,t_w)) &= z^{(i-1)}((u,1),(w,t_w)) + \gamma_{i-1} \\
z^{(i)}((u,t_u),(w,1)) &= z^{(i-1)}((u,t_u),(w,1)) + \gamma_{i-1}
\end{aligned}\right\} \tag{34}$$

As a result

$$\begin{aligned}
&z^{(i)}((u,1),(w,1)) \\
&\quad= -\gamma_{i-1} + z^{(i-1)}((u,1),(w,1)) \\
&\quad\le -\gamma_{i-1} + \sum\left\{z^{(i-1)}((u,t_u),(w,t_w)) : t_u \neq t_w \in K\setminus\{1\}\right\} \\
&\quad= \sum\left\{z^{(i)}((u,t_u),(w,t_w)) : t_u \neq t_w \in K\setminus\{1\}\right\}.
\end{aligned}$$

Thus, we set $i \leftarrow i+1$, pick a non-zero value $z^{(i-1)}((u,t_u),(w,t_w)) > 0, t_u \neq t_w \in K\setminus\{1\}$, and repeat (34), until $z^{(i)}((u,1),(w,1)) = 0$. Finally, $(x^{(i)}, y^{(i)}) = (x^q, y^q = 0)$, completing the proof of Theorem 5.

4.2.2 Positive edge variable

In this whole section, we prove that for $q = (x^q, y^q_{uw}) \in LQ1(G,k)$ with $y^q_{uw} > 0$, there is $z \in ELP1(G,k)$ such that $Proj(z) = q = (x^q, y^q_{uw})$:

Theorem 7. *For $k \ge 3$, let $G = uw$ be the edge graph. Let $(x^q, y^q_{uw}) \in LQ1(G,k)$ with $y^q_{uw} > 0$. Then, there is a non-negative solution z to (9) satisfying (11)-(12) with $(x,y) = (x^q, y^q)$.*

First, we will define non-negative variables $z((u,t),(w,t)) = \delta_t y^q_{uw}$ for $t = 1, ..., k$, where $(\delta_t)^k_{t=1}$ are non-negative coefficients satisfying $\sum_{t=1}^k \delta_t = 1$. If $y^q_{uw} = 1$, we can define the remaining $z((u,t_u),(w,t_w)) = 0$ with $t_u \neq t_w$ and $Proj(z) = (x^q, y^q_{uw})$. The rest of the proof focuses on the case $y^q_{uw} < 1$. For this case we will show that $x'_{ut} = x^q_{ut} - z((u,t),(w,t)), x'_{wt} = x^q_{wt} - z((u,t),(w,t))$ for $t = 1,...,k$ and $y'_{uw} = 0$ can be scaled to satisfy

$$q^0 = (x^0, y^0_{uw} = 0) = \frac{(x', y'_{uw} = 0)}{1 - y^q_{uw}} \in LQ1(G,k).$$

By Theorem 5, there will exist $z^0 \geq 0$ such that $Proj(z^0) = (x^0, y^0_{uw} = 0)$. Then, we will scale z^0 back and define

$$z((u,t_u),(w,t_w)) = z^0((u,t_u),(w,t_w)) \cdot (1 - y^q_{uw})$$

for $t_u \neq t_w$.

To prove Theorem 7 we need the following proposition:

Proposition 8. *Let $q = (x^q, y^q_{uw}) \in LQ1(G,k)$. Then,*

$$\sum_{t=1}^{k} \min\{x^q_{ut}, x^q_{wt}\} \geq y^q_{uw}.$$

Proof. Let $S = \{t \in K : x^q_{ut} \geq x^q_{wt}\}$. Then, the partitioning equation (2) for $v = u$ and a generalized arc inequality in (6) imply

$$\begin{aligned}
\sum_{t=1}^{k} \min\{x^q_{ut}, x^q_{wt}\} &= \sum_{t \in K \setminus S} x^q_{ut} + \sum_{t \in S} x^q_{wt} \\
&= \left(1 - \sum_{t \in S} x^q_{ut}\right) + \sum_{t \in S} x^q_{wt} = \left(1 - \sum_{t \in S} x^q_{ut} + \sum_{t \in S} x^q_{wt}\right) \\
&\geq y^q_{uw},
\end{aligned}$$

completing the proof of the proposition.

□

If $y^q_{uw} = 1$, Proposition 8 can be used to define $z \geq 0$ with $Proj(z) = (x^q, y^q_{uw})$ by

$$z((u,t),(w,t)) = x^q_{ut} = x^q_{wt} \text{ for } t \in K$$
$$z((u,t_u),(w,t_w)) = 0 \text{ for } t_u \neq t_w \in K.$$

Assuming $y^q_{uw} < 1$, Proposition 8 implies the following lemma:

Lemma 9. *Let $k \geq 2$ and let $q = (x^q, y^q_{uw}) \in LQ1(G = uw, k)$ with $0 < y^q_{uw} < 1$. Then, there exists $\delta = (\delta_1, \delta_2, ..., \delta_k)$ with $\delta_t \geq 0, t = 1,...,k$, and $\sum_{t=1}^{k} \delta_t = 1$,such that $x'_{ut} = x^q_{ut} - \delta_t y^q_{uw} \geq 0, x'_{wt} = x^q_{wt} - \delta_t y^q_{uw} \geq 0$ for $t = 1,...,k$ and $y'_{uw} = 0$ satisfy*

$$q^0 = (x^0, y^0_{uw} = 0) = \frac{(x', y'_{uw} = 0)}{1 - y^q_{uw}} \in LQ1(G,k).$$

Proof. By Proposition 8 there exists $\delta = (\delta_t \geq 0 : t = 1,...,k)$ with $\sum_{t=1}^{k} \delta_t = 1$ such that

$$\left.\begin{aligned} x'_{ut} = x^q_{ut} - \delta_t y^q_{uw} \geq 0 \\ x'_{wt} = x^q_{wt} - \delta_t y^q_{uw} \geq 0 \end{aligned}\right\} \text{ for } t \in K. \tag{35}$$

Then, the partitioning equations (2) and the generalized arc inequalities (6) at

$$(x^0, y^0_{uw} = 0) = \frac{(x', y'_{uw} = 0)}{1 - y^q_{uw}}$$

are implied by

$$\begin{aligned}
\sum_{t\in K} x'_{ut} &= \sum_{t\in K} (x^q_{ut} - \delta_t y^q_{uw}) = \sum_{t\in K} x^q_{ut} - \sum_{t\in K} \delta_t y^q_{uw} = 1 - y^q_{uw} \\
\sum_{t\in K} x'_{wt} &= \sum_{t\in K} (x^q_{wt} - \delta_t y^q_{uw}) = \sum_{t\in K} x^q_{wt} - \sum_{t\in K} \delta_t y^q_{uw} = 1 - y^q_{uw}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{t\in S} x'_{ut} - \sum_{t\in S} x'_{wt} &= \sum_{t\in S} (x^q_{ut} - \delta_t y^q_{uw} - x^q_{wt} + \delta_t y^q_{uw}) = \sum_{t\in S} (x^q_{ut} - x^q_{wt}) \le 1 - y^q_{uw} \\
-\sum_{t\in S} x'_{ut} + \sum_{t\in S} x'_{wt} &= \sum_{t\in S} (-x^q_{ut} + \delta_t y^q_{uw} + x^q_{wt} - \delta_t y^q_{uw}) = \sum_{t\in S} (-x^q_{ut} + x^q_{wt}) \le 1 - y^q_{uw}.
\end{aligned}$$

An edge inequality (3) may not hold at $(x^0, y^0_{uw} = 0)$. That is,

$$\frac{x'_{ut} + x'_{wt}}{1 - y^q_{uw}} = x^0_{ut} + x^0_{wt} = x^0_{ut} + x^0_{wt} - y^0_{uw} \le 1,$$

or equivalently,

$$x'_{ut} + x'_{wt} \le 1 - y^q_{uw},$$

may not hold. However, we will adjust δ for $(x^0, y^0_{uw} = 0)$ to satisfy all the edge inequalities. To this end, without loss of generality, we may assume that

$$x'_{ut} + x'_{wt} \ge 1 - y^q_{uw} \quad \text{for} \quad t \le r, \tag{36}$$

$$x'_{ut} + x'_{wt} < 1 - y^q_{uw} \quad \text{for} \quad t > r. \tag{37}$$

If there is $t > r$ such that $\delta_t > 0$, we may decrease $\delta_t \to \delta_t - \theta$ increasing the left-hand side of the scaled edge inequality in (37)

$$x'_{ut} + x'_{wt} \le 1 - y^q_{uw}$$

while decreasing the left-hand side of a violated scaled edge inequality in (36), say for $t = 1$, by $\delta_1 \to \delta_1 + \theta$ (all subject to (35) for (2) and (6)).

In order to construct $(x^0, y^0_{uw} = 0)$ satisfying all the edge inequalities (3), we only need to show that, if $\delta_t = 0$ for all $t > r$, there is no violated scaled edge inequality in (36). Assume for a contradiction that $\delta_t = 0$ for all $t > r$ and

$$x'_{u1} + x'_{w1} > 1 - y^q_{uw}.$$

Then, partitioning equations (2) imply that

$$\begin{aligned}
2(1 - y^q_{uw}) &= -2y^q_{uw} + 1 + 1 \\
&\ge -2y^q_{uw} + \sum_{t=1}^{r} x^q_{ut} + \sum_{t=1}^{r} x^q_{wt} = -2\sum_{t=1}^{r} \delta_t y^q_{uw} + \sum_{t=1}^{r} x^q_{ut} + \sum_{t=1}^{r} x^q_{wt} \\
&= \sum_{t=1}^{r} x^q_{ut} - \sum_{t=1}^{r} \delta_t y^q_{uw} + \sum_{t=1}^{r} x^q_{wt} - \sum_{t=1}^{r} \delta_t y^q_{uw} = \sum_{t=1}^{r} (x'_{ut} + x'_{wt}) \\
&> r \cdot (1 - y^q_{uw}),
\end{aligned}$$

which is a contradiction for $r \ge 2$. If $r = 1$, then $\delta_1 = 1$ and

$$(x^q_{u1} - y^q_{uw}) + (x^q_{w1} - y^q_{uw}) = (x^q_{u1} - \delta_1 y^q_{uw}) + (x^q_{w1} - \delta_1 y^q_{uw}) > 1 - y^q_{uw},$$

which would imply a contradiction $x^q_{u1} + x^q_{w1} - y^q_{uw} > 1$ violating an edge inequality at (x^q, y^q_{uw}), completing the proof of the lemma. □

From Lemma 9, we conclude Theorem 7.

Acknowledgement. We are grateful to the anonymous referees for suggesting different approaches to prove theorems in this paper. The authors thank Dabeen Lee for fruitful discussions. He first identified the generalized arc inequalities in [10].

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A Alternative ways to prove the integrality of $ELP1$ on a tree

In this section, we outline how Theorem 4 can be proved using four different approaches.

Theorem 4 Let G be a tree. Then, $ELP1(G,k) = EP1(G,k)$.

A.1 A straightforward proof

In this section we prove that when G is a tree, any fractional solution $z \in ELP1(G,k)$ can be obtained as the average of two feasible solutions $z^1, z^2 \in ELP1(G,k)$, *i.e.*, $z = 0.5z^1 + 0.5z^2$.

Proof. Let $z \in ELP1(G,k)$ be a fractional solution with a variable $z\left((u,t_u),(w,t_w)\right)$ that has fractional value f_0. By (9), there must be another fractional variable $z\left((u,\bar{t}_u),(w,\bar{t}_w)\right) = \bar{f}_0$ that is also fractional (say with value $\bar{f}_0$) such that at least one of the following holds: $t_u \neq \bar{t}_u, t_w \neq \bar{t}_w$. Observe that both $((u,t_u),(w,t_w))$ and $((u,\bar{t}_u),(w,\bar{t}_w))$ are edges in G^k from the bipartite graph corresponding to the edge (u,w) in G. Define $u_0 = u$ and let $w_0 = w$.

We now extend each of the two fractional variables $z((u,t_{u_0}),(w,t_{w_0}))$ and $z((u,\bar{t}_{u_0}),(w,\bar{t}_{w_0}))$ along two paths in G^k that either end at a shared node in G^k or at nodes in G^k corresponding to a leaf node in G. The extension is done iteratively on the w side as follows.

1. Set $j = 0$.
2. If either $t_{w_j} = \bar{t}_{w_j}$ or w_j is a leaf node in G, stop. If $t_{w_j} \neq \bar{t}_{w_j}$ and w_j is not a leaf node in G, by (9)-(10) there must exist two fractional variables $z((w_j,t_{w_j}),(w_{j+1},t_{w_{j+1}}))$ and $z((w_j,\bar{t}_{w_j}),(w_{j+1},\bar{t}_{w_{j+1}}))$.
3. Set $j = j+1$. Go to step 2.

Observe that because G is a tree, the iterative procedure must stop at step 2 for some value of j, say j^*.

The extension is done iteratively on the u side in a similar manner as follows.

1. Set $j = 0$.
2. If either $t_{u_j} = \bar{t}_{u_j}$ or u_j is a leaf node in G, stop. If $t_{u_j} \neq \bar{t}_{u_j}$ and u_j is not a leaf node in G, by (9)-(10) there must exist two fractional variables $z((u_{j-1},t_{u_{j-1}}),(u_j,t_{u_j}))$ and $z((u_{j-1},\bar{t}_{u_{j-1}}),(u_j,\bar{t}_{u_j}))$.
3. Set $j = j-1$. Go to step 2.

Once again because G is a tree, the iterative procedure must stop at step 2 for some value of j, say $-\hat{j}$.

Let P be the path in G^k defined by $((u_{-\hat{j}},t_{u_{-\hat{j}}}),(u_{-\hat{j}+1},t_{u_{-\hat{j}+1}}))$, ..., $((w_{j^*-1},t_{w_{j^*-1}}),(w_{j^*},t_{w_{j^*}}))$. Let $\bar{P}$ be the path in G^k defined by $((u_{-\hat{j}},\bar{t}_{u_{-\hat{j}}}),(u_{-\hat{j}+1},\bar{t}_{u_{-\hat{j}+1}}))$, ..., $((w_{j^*-1},\bar{t}_{w_{j^*-1}}),(w_{j^*},\bar{t}_{w_{j^*}}))$. Observe that both P and $\bar{P}$ correspond to the same path in G. Observe that z has fractional values on all edges in G^k corresponding to both P and $\bar{P}$. Figure 7 illustrates an example of path $u_{-\hat{j}} - ... - w_{j^*}$ where the extension of u is ending at a shared node $t_{u_{-\hat{j}}} = \bar{t}_{u_{-\hat{j}}}$ and the extension of w is ending with a leaf node $l = w_{j^*}$. By the 3-step procedure above, the paths P and $\bar{P}$ may branch at each node of the paths and extend to trees T and $\bar{T}$ in the depth-first search manner. Each branch ends up with a common node like $t_{-\hat{j}} = \bar{t}_{-\hat{j}}$ on the left or a leaf node like $(l,t_l),(l,\bar{t}_l)$ on the right in Figure 7.

We now construct two feasible solutions z^1 and z^2 in $ELP1(G,k)$ as follows. For all edges $e \in E^k \setminus \{T \cup \bar{T}\}$ set $z^1(e) = z^2(e) = z(e)$. For $e \in T$, set $z^1(e) = z(e) + \varepsilon$, $z^2(e) = z(e) - \varepsilon$. For $e \in \bar{T}$, set $z^1(e) = z(e) - \varepsilon$, $z^2(e) = z(e) + \varepsilon$. Observe that for $\varepsilon > 0$ and small enough both z^1 and z^2 are in $ELP1(G,k)$. We also have $z = 0.5z^1 + 0.5z^2$.

This implies that no fractional point in $ELP1(G,k)$ can be an extreme point. Thus $ELP1(G,k)$ has only integer extreme points.

□

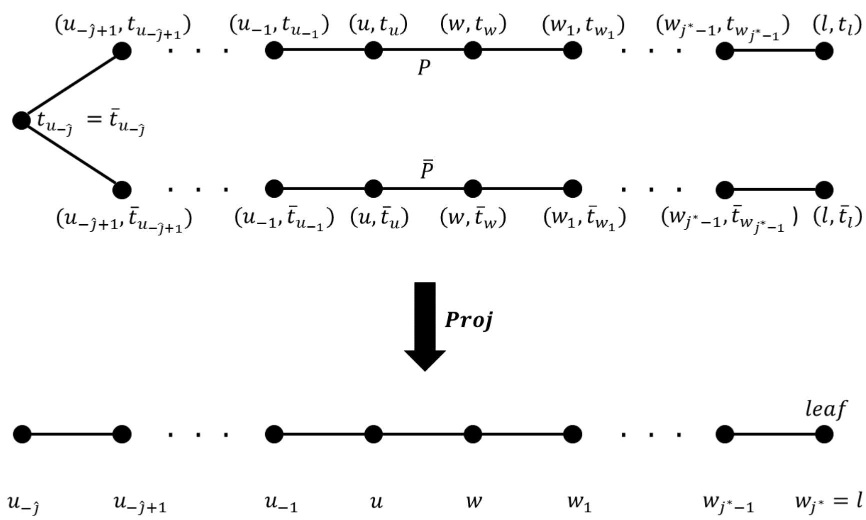

Figure 7: A fractional solution cannot be an extreme point over a tree

A.2 Gluing the graph partition polyhedra on single edges

Along with Section 2.1, Section 4 shows that $LQ1$ is equivalent to $ELP1$. In particular, $LQ1$ is integral on a single edge. Then, the following propositions imply that $LQ1$ is tight on a tree and therefore $ELP1$ is. Let $P1(G,k) \subseteq LQ1(G,k)$ (or simply $P1(G)$) denote the convex hull of the integer vectors in $LQ1(G,k)$.

Proposition 10. *Let G_1 and G_2 be two subgraphs of a graph G with exactly one common node partitioning the edge set; i.e., $E(G_1) \cup E(G_2) = E(G)$, $E(G_1) \cap E(G_2) = \emptyset$, $V(G_1) \cup V(G_2) = V(G)$ and $|V(G_1) \cap V(G_2)| = 1$. Constraints determining $P1(G_1)$ and ones determining $P1(G_2)$ together describe $P1(G)$ completely.*

Such a node without which the graph becomes disconnected is called a *cut node*. A *block* of a graph is a maximal connected subgraph with no cut node–a subgraph with as many edges as possible and no cut node. So a block is either the edge graph or is a graph which contains a cycle. We can gather the constraints of blocks together to build up a complete set of constraints for $P1(G,k)$ of the whole graph.

Proposition 11. *Let $G_1, ..., G_\omega$ be the blocks of a graph G. All constraints determining $P1(G_i)$ for $i = 1, ..., \omega$ completely describe $P1(G)$.*

Each block of a tree is an edge. The propositions along with the integrality of $LQ1$ on a single edge immediately imply the fact that $LQ1$ (and therefore $ELP1$) is tight on a tree.

A.3 Projected faces property

In this section, we construct an auxiliary graph and show by the projected faces property that the auxiliary graph formulation of the graph partition problem is tight on a tree. A polyhedron $P \subseteq \mathbb{R}^n \times \mathbb{R}^d$, together with the projection map $p_x : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ has the *projected faces property* if every face $F \subseteq \mathbb{R}^n \times \mathbb{R}^d$ is projected to a face of the polytope $p_x(P)$. That is, for every face $F \subseteq \mathbb{R}^n \times \mathbb{R}^d$ of the polytope P the projection $p_x(F) \subseteq \mathbb{R}^n$ is a face of the polytope $p_x(P)$. For the details of the projected faces property, we refer the reader to Conforti and Pashkovich [5].

In his dissertation, Margot [12] showed the following:

Theorem 12. *Given systems $A_1(z_1^x, z_1) \le b_1$ and $A_2(z_2^x, z_2) \le b_2$ defining the polytopes $P_1 \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{d_1}$ and $P_2 \subseteq \mathbb{R}^{n_2} \times \mathbb{R}^{d_2}$ and a function $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^n \cup \{\infty\}$, let $A_3(x, z_1^x, z_2^x) \le b_3$ define the following* *polytope*

$$P_3 := \operatorname{conv}\left\{(x, z_1^x, z_2^x) : z_1^x \in \operatorname{vert}\left(p_{z_1^x}(P_1)\right), z_2^x \in \operatorname{vert}\left(p_{z_2^x}(P_2)\right), x = f\left(z_1^x, z_2^x\right) \text{ and } x \neq \infty\right\}.$$

Let Q be the polyhedron defined by the three systems,

$$\begin{aligned} Q := \big\{&(x, z_1^x, z_1, z_2^x, z_2) \in \mathbb{R}^n \times \mathbb{R}^{n_1} \times \mathbb{R}^{d_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{d_2} : \\ &A_1(z_1^x, z_1) \leq b_1, A_2(z_2^x, z_2) \leq b_2, A_3(x, z_1^x, z_2^x) \leq b_3\big\}, \end{aligned}$$

and let P be the convex hull of the following set of points $(x, z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ such that $(z_1^x, z_1) \in \operatorname{vert}(P_1)$, $(z_2^x, z_2) \in \operatorname{vert}(P_2)$ and $x = f(z_1^x, z_2^x) \neq \infty$ for some $z_1^x \in \mathbb{R}^{n_1}$, $z_2^x \in \mathbb{R}^{n_2}$. If both $(P_1, p_{z_1^x})$ and $(P_2, p_{z_2^x})$ have the PF-property and every vertex of P_3 projects into a vertex of $p_x(P_3)$, then $P = p_{x,z_1,z_2}(Q)$.

For a graph $G = (V, E)$, we construct an *auxiliary graph* $H = (V_H, E_H)$ defined by

$$\begin{aligned} V_H &= V \cup (V \times K), \\ E_H &= \Big\{\{v, (v,t)\} : v \in V \text{ and } t \in K\Big\} \cup \Big\{\{(u, t_u), (w, t_w)\} : \{u, w\} \in E \text{ and } t_u, t_w \in K\Big\}. \end{aligned}$$

Considering the direction (u, v) of a single edge $uv \in E$, one unit of flow from u to v goes through the auxiliary graph H_{uv}. The flow model on a single arc (u, v) is formulated by flow conservation constraints with flow variables $z(u, (u, t))$, $z((u, t_u), (v, t_v))$ and $z((u, t), v)$ for $t, t_u, t_v \in K$, and it is easy to see that the system is totally unimodular. Over a 2-path $u - v - w$, we glue H_{uv} and H_{vw} at node v such that

$$x_{vt} = z((v, t), v) = z(v, (v, t)) \text{ for } t \in K. \tag{38}$$

In Theorem 12, we set P_1 and P_2 to be the polytopes defined over H_{uv} and H_{vw} employing variable vectors

$$\begin{aligned} z_1^x &= (z((v, t), v) : t \in K), \\ z_2^x &= (z(v, (v, t)) : t \in K), \\ z_1 &= (z(u, (u, t_u)), z((u, t_u), (v, t_v)) : t_u, t_v \in K), \\ z_2 &= (z((v, t_v), (w, t_w)), z((w, t_w), w) : t_v, t_w \in K), \end{aligned}$$

and we replace $A_3(x, z_1^x, z_2^x) \leq b_3$ by (38) along with

$$\sum_{t \in K} x_{vt} = 1$$

which defines a $(k-1)$-simplex along with the non-negativity constraints. Since any subset of the vertices of a simplex forms a simplex, both $(P_1, p_{z_1^x})$ and $(P_2, p_{z_2^x})$ have the PF-property. Every vertex of P_3 projects into a vertex of $p_x(P_3)$, and $p_{x,z_1,z_2}(Q)$ is the integer hull P. Such a gluing iteratively maintains integrality, and proves the integrality of the auxiliary graph formulation on a tree.

A.4 Path-flow model of the extended graph formulation

The auxiliary graph formulation in Section A.3 is related to the path-flow model of an arborescence introduced by Conforti, Cornuejols and Zambelli [4]. This connection allows us to interpret our extended graph formulation from the alternative perspective of path-flow.

Consider the auxiliary graph H of a tree G rooted at $r \in V$. Every tree edge in E is assume to be directed outward from the root r. For every non-root node $v \in V \setminus \{r\}$, there is a unique path from r to v in the tree. We send the one-unit of path-flow from r to v in H with capacities x_{rt} on $(r, (r, t))$ at the beginning, $z((u, t_u), (w, t_w))$ on arcs (u, w) in the middle, and x_{vt} on $((u, v), v)$ at the end. Note that no fraction of the unit flow can be derailed out of the sub-graph of H lifted from the unique path. The capacities are fully saturated because of (2) and (9). Note that (10) and (12) are flow conservation constraints. This path-flow perspective allows us to obtain an alternative understanding of the proof of Theorem 4.

B Exponentially many GAIs are required for $LQ1(G,k)$

In this section, we show Proposition 2. Shim and Johnson [15] identified a minimal description of the cyclic group facet polytope. They provided a fractional point that is violated by only one of the inequalities. In the same way, Chopra, Shim and Steffy [3] identified a minimal description of the knapsack facet polytope. Following the way, we show Proposition 2.

Proposition 2 Let $G=(V,E)$ be a graph, and let $k\geq 3$. Then the following system is a minimal description of $LQ1(G,k)$:

$$(x,y)\geq 0, \tag{39}$$

$$\sum_{t=1}^{k} x_{vt}=1 \qquad \text{for all nodes } v, \tag{40}$$

$$x_{ut}+x_{wt}-y_{uw}\leq 1 \qquad \text{for all edges } uw \text{ and all } t=1,\dots,k, \tag{41}$$

$$\left.\begin{array}{r}\sum_{t\in S}(x_{ut}-x_{wt})+y_{uw}\leq 1\\ -\sum_{t\in S}(x_{ut}-x_{wt})+y_{uw}\leq 1\end{array}\right\} \qquad \text{for all edges } uw \text{ and all } S\subseteq\{1,\dots,k-1\}. \tag{42}$$

Proof. [1] We first prove the following claim:

Claim 1. Each constraint of type (39)–(41) *is non-redundant.*

Proof of Claim. We first show that $x_{vt}\geq 0$ is non-redundant for every $v\in V$ and $t\in\{1,\dots,k\}$. To this end, take $v\in V$ and $t\in\{1,\dots,k\}$. Let ε be a positive number less than $\frac{1}{k}$, and let $(\bar{x}^1,\bar{y}^1)$ be the vector defined as follows:

$$\bar{x}^1_{us}=\begin{cases}-\varepsilon & \text{for } u=v \text{ and } s=t\\ \frac{1+\varepsilon}{k-1} & \text{for } u=v \text{ and } s\neq t\\ \frac{1}{k} & \text{for } u\neq v\end{cases}, \qquad \bar{y}^1_e=0 \text{ for all edges } e.$$

Notice that $(\bar{x}^1,\bar{y}^1)$ satisfies all constraints but $x_{vt}\geq 0$ as $-\bar{x}^1_{vt}=-\varepsilon$, implying in turn that $x_{vt}\geq 0$ is non-redundant. It is also easy to show that $y_{uw}\geq 0$ is non-redundant for every $uw\in E$. Let $(\bar{x}^2,\bar{y}^2)$ be the vector defined as follows:

$$\bar{x}^2_{vt}=\frac{1}{k} \text{ for all nodes } v \text{ and all } t, \qquad \bar{y}^2_e=\begin{cases}-\frac{1}{k} & \text{for } e=uw\\ 0 & \text{for } e\neq uw\end{cases}.$$

As $k\geq 3$, it can be observed that $(\bar{x}^2,\bar{y}^2)$ satisfies all constraints but $y_{uw}\geq 0$, which implies that $y_{uw}\geq 0$ is non-redundant.

Next we show that $\sum_{t=1}^{k}x_{vt}=1$ is non-redundant for every $v\in V$. To this end, take $v\in V$. Let $(\bar{x}^3,\bar{y}^3)$ be the vector defined as follows:

$$\bar{x}^3_{ut}=\begin{cases}0 & \text{for } u=v\\ \frac{1}{k} & \text{for } u\neq v\end{cases}, \qquad \bar{y}^3_e=0 \text{ for all edges } e.$$

One can easily check that $(\bar{x}^3,\bar{y}^3)$ satisfies all constraints but $\sum_{t=1}^{k}x_{vt}=1$, and therefore, constraint $\sum_{t=1}^{k}x_{vt}=1$ is non-redundant.

[1] This proof is done by Dabeen Lee at Carnegie Mellon University.

Lastly, we prove that each inequality of type (41) is non-redundant. To this end, take an edge $uw \in E$ and $t \in\subseteq \{1, \ldots, k\}$. Let $(\bar{x}^4, \bar{y}^4)$ be the vector defined as follows:

$$\bar{x}^4_{vs} = \begin{cases} 1 & \text{for } v \in \{u, w\} \text{ and } s = t \\ 0 & \text{for } v \in \{u, w\} \text{ and } s \neq t \\ \frac{1}{k} & \text{for } v \notin \{u, w\} \end{cases}, \qquad \bar{y}_e = \begin{cases} 0 & \text{for } e = uw \\ \frac{1}{k} & \text{for } e \neq uw \end{cases}.$$

It can be observed that that $(\bar{x}^4, \bar{y}^4)$ satisfies all constraints but $x_{ut} + x_{wt} - y_{uw} \leq 1$, so $x_{ut} + x_{wt} - y_{uw} \leq 1$ is non-redundant. ♢

What remains is to show that each inequality of type (42) is non-redundant.

Claim 2. For every edge $uw \in E$ and $S \subseteq \{1, \ldots, k-1\}$, both $\sum_{t\in S}(x_{ut} - x_{wt}) + y_{uw} \leq 1$ and $-\sum_{t\in S}(x_{ut} - x_{wt}) + y_{uw} \leq 1$ are non-redundant.

Proof of Claim. Let $(\bar{x}, \bar{y})$ be the vector defined as follows:

$$\bar{x}_{vt} = \begin{cases} \frac{1}{|S|} & \text{for } v = u \text{ and } t \in S \\ 0 & \text{for } v = u \text{ and } t \notin S \\ 0 & \text{for } v = w \text{ and } t \in S \\ \frac{1}{k-|S|} & \text{for } v = w \text{ and } t \notin S \\ \frac{1}{k} & \text{for } v \notin \{u, w\} \end{cases}, \qquad \bar{y}_e = \frac{1}{k} \text{ for all edges } e.$$

It can be easily observed that $(\bar{x}, \bar{y})$ satisfies (39)–(41). We would like to show that $(\bar{x}, \bar{y})$ satisfies all constraints but

$$\sum_{t\in S}(x_{ut} - x_{wt}) + y_{uw} \leq 1. \tag{43}$$

Notice that

$$\sum_{t\in S}(\bar{x}_{ut} - \bar{x}_{wt}) + \bar{y}_{uw} = 1 + \frac{1}{k} \not\leq 1,$$

implying in turn that $(\bar{x}, \bar{y})$ violates (43). It is easy to show that $(\bar{x}, \bar{y})$ satisfies the other constraints in (39)–(42).

Let $(\bar{x}', \bar{y}')$ be the vector defined as follows:

$$\bar{x}'_{vt} = \begin{cases} 0 & \text{for } v = u \text{ and } t \in S \\ \frac{1}{k-|S|} & \text{for } v = u \text{ and } t \notin S \\ \frac{1}{|S|} & \text{for } v = w \text{ and } t \in S \\ 0 & \text{for } v = w \text{ and } t \notin S \\ \frac{1}{k} & \text{for } v \notin \{u, w\} \end{cases}, \qquad \bar{y}'_e = \frac{1}{k} \text{ for all edges } e.$$

Notice that $\bar{x}'$ is obtained from $\bar{x}$ after switching u and w. This in turn implies that $(\bar{x}', \bar{y}')$ satisfies all constraints but

$$-\sum_{t\in S}(x_{ut} - x_{wt}) + y_{uw} \leq 1, \tag{44}$$

as required. ♢

□

Then we can briefly prove the following proposition:

Proposition 13. *The generalized arc inequalities are facet defining for $LQ1(G, k)$.*

Proof. By Proposition 2,

$$\left.\begin{array}{r}\sum_{t\in S}(x_{ut}-x_{wt})+y_{uw}\leq 1\\ -\sum_{t\in S}(x_{ut}-x_{wt})+y_{uw}\leq 1\end{array}\right\}$$

are facet defining for all edges uw and $S \subseteq \{1,\ldots,k-1\}$. For a subset S of $\{1,\ldots,k\}$ containing k, the complement of S, denoted $\bar{S}$, is contained in $\{1,\ldots,k-1\}$. Notice that $\sum_{t\in S}(x_{ut}-x_{wt})+y_{uw}\leq 1$ is equivalent to $-\sum_{t\in\bar{S}}(x_{ut}-x_{wt})+y_{uw}\leq 1$ and that $-\sum_{t\in S}(x_{ut}-x_{wt})+y_{uw}\leq 1$ is equivalent to $\sum_{t\in\bar{S}}(x_{ut}-x_{wt})+y_{uw}\leq 1$. Therefore, all generalized arc inequalities are facet defining. □